Solution to HW 10

Leon Li

ylli@math.cuhk.edu.hk

MATH 2020B HW10

Due Date: May 1, 2020 (12:00 noon)

Thomas' Calculus (12th Ed.)

§16.7: 4, 8, 16, 26

§16.8: 10, 14, 16, 20, 23, 27

§16.7

Using Stokes' Theorem to Find Line Integrals

In Exercises 1–6, use the surface integral in Stokes' Theorem to calculate the circulation of the field **F** around the curve C in the indicated direction.

4. $\mathbf{F} = (y^2 + z^2)\mathbf{i} + (x^2 + z^2)\mathbf{j} + (x^2 + y^2)\mathbf{k}$

C: The boundary of the triangle cut from the plane $x + y + z = 1$ by the first octant, counterclockwise when viewed from above

Sol) $\vec{F}(x,y,z) = (y^2+z^2)\vec{i} + (x^2+z^2)\vec{j} + (x^2+y^2)\vec{k}.$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+z^2 & x^2+z^2 & x^2+y^2 \end{vmatrix} = (2y-2z)\vec{i} - (2x+2z)\vec{j} + (2x+2y)\vec{k}.$$

Let $f(x,y,z) = x+y+z$; $\nabla f(x,y,z) = \vec{i}+\vec{j}+\vec{k}$; $\vec{n} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{3}}(\vec{i}+\vec{j}+\vec{k}).$

$$\operatorname{curl}\vec{F}\cdot\vec{n} = \frac{1}{\sqrt{3}}\big((2y-2z)-(2x+2z)+(2x+2y)\big) = 0.$$

$$\therefore \oint_C \vec{F}\cdot d\vec{r} = \iint_S 0\, d\sigma = 0 \,/\!/$$

Flux of the Curl

8. Let **n** be the outer unit normal (normal away from the origin) of the parabolic shell

$$S: \quad 4x^2 + y + z^2 = 4, \qquad y \geq 0,$$

and let

$$\mathbf{F} = \left(-z + \frac{1}{2+x}\right)\mathbf{i} + (\tan^{-1} y)\mathbf{j} + \left(x + \frac{1}{4+z}\right)\mathbf{k}.$$

Find the value of

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma.$$

Sol) $\vec{F}(x,y,z) = \left(-z + \frac{1}{2+x}\right)\vec{i} + (\tan^{-1} y)\vec{j} + \left(x + \frac{1}{4+z}\right)\vec{k}.$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z + \frac{1}{2+x} & \tan^{-1} y & x + \frac{1}{4+z} \end{vmatrix} = -(1+1)\vec{j} = -2\vec{j}.$$

Let $f(x,y,z) = 4x^2 + y + z^2$; $\nabla f(x,y,z) = 8x\vec{i} + \vec{j} + 2z\vec{k}$; $|\nabla f \cdot \vec{j}| = 1.$

$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \vec{j}|} dA = |\nabla f|\, dA$; $\vec{n} = \frac{\nabla f}{|\nabla f|} = \frac{1}{|\nabla f|}(8x\vec{i} + \vec{j} + 2z\vec{k}).$

$\text{curl}\,\vec{F} \cdot \vec{n}\, d\sigma = \left((-2j) \cdot \frac{1}{|\nabla f|}(8x\vec{i} + \vec{j} + 2z\vec{k})\right)|\nabla f|\, dA = -2\, dA$

$\therefore \iint_S \text{curl}\,\vec{F} \cdot \vec{n}\, d\sigma = \iint_R -2\, dA$, where $R = \{(x,z) \in \mathbb{R}^2 \mid 4x^2 + z^2 \leq 4\}$

Let $\begin{cases} x = \rho\cos\theta \\ z = 2\rho\sin\theta \end{cases}$, where $\rho \geq 0$ and $0 \leq \theta < 2\pi$. Then $4x^2 + z^2 \leq 4 \Leftrightarrow \rho \leq 1.$

$\therefore R = \{(\rho,\theta) \in [0,+\infty) \times [0,2\pi) \mid \rho \leq 1\}$

$$\frac{\partial(x,z)}{\partial(\rho,\theta)} = \begin{vmatrix} \cos\theta & -\rho\sin\theta \\ 2\sin\theta & 2\rho\cos\theta \end{vmatrix} = 2\rho; \quad \left|\frac{\partial(x,z)}{\partial(\rho,\theta)}\right| = 2\rho$$

$$\therefore \iint_R -2\, dA = -2\int_0^{2\pi}\int_0^1 2\rho\, d\rho\, d\theta = -2 \cdot 2\pi \cdot 1 = -4\pi_{/\!/}$$

Stokes' Theorem for Parametrized Surfaces

In Exercises 13–18, use the surface integral in Stokes' Theorem to calculate the flux of the curl of the field **F** across the surface S in the direction of the outward unit normal **n**.

16. $\mathbf{F} = (x - y)\mathbf{i} + (y - z)\mathbf{j} + (z - x)\mathbf{k}$

S: $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + (5 - r)\mathbf{k}$, $0 \le r \le 5$, $0 \le \theta \le 2\pi$

Sol) $\vec{F}(x,y,z) = (x-y)\vec{i} + (y-z)\vec{j} + (z-x)\vec{k}$.

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-y & y-z & z-x \end{vmatrix} = (0-(-1))\vec{i} - (-1-0)\vec{j} + (0-(-1))\vec{k} = \vec{i} + \vec{j} + \vec{k}.$$

$\vec{r}(r,\theta) = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} + (5-r)\vec{k}$, where $0 \le r \le 5$ and $0 \le \theta < 2\pi$.

$\vec{r}_r(r,\theta) = \cos\theta\,\vec{i} + \sin\theta\,\vec{j} - \vec{k}$; $\vec{r}_\theta(r,\theta) = -r\sin\theta\,\vec{i} + r\cos\theta\,\vec{j}$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & -1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (0 - r\cos\theta)\vec{i} - (0 - r\sin\theta)\vec{j} + (r\cos^2\theta + r\sin^2\theta)\vec{k}$$

$$= -r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} + r\vec{k}.$$

$\operatorname{curl}\vec{F}\cdot\vec{n}\,d\sigma = \operatorname{curl}\vec{F}\cdot(\vec{r}_r \times \vec{r}_\theta)\,dr\,d\theta = (\vec{i} + \vec{j} + \vec{k})\cdot(-r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} + r\vec{k})\,dr\,d\theta$

$= (-r\cos\theta + r\sin\theta + r)\,dr\,d\theta$

$\therefore \iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\,d\sigma = \int_0^{2\pi}\int_0^5 (-r\cos\theta + r\sin\theta + r)\,dr\,d\theta = \left(\int_0^{2\pi}(-\cos\theta + \sin\theta + 1)\,d\theta\right)\cdot\left(\int_0^5 r\,dr\right)$

$= \left[-\sin\theta - \cos\theta + \theta\right]_0^{2\pi}\left[\frac{r^2}{2}\right]_0^5 = 2\pi\cdot\frac{25}{2} = 25\pi$ //

Theory and Examples

26. **Zero curl, yet field not conservative** Show that the curl of

$$\mathbf{F} = \frac{-y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j} + z\mathbf{k}$$

is zero but that

$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$

is not zero if C is the circle $x^2 + y^2 = 1$ in the xy-plane. (Theorem 7 does not apply here because the domain of $\mathbf{F}$ is not simply connected. The field $\mathbf{F}$ is not defined along the z-axis so there is no way to contract C to a point without leaving the domain of $\mathbf{F}$.)

Sol) $\vec{F}(x,y,z) = \left(\frac{-y}{x^2+y^2}\right)\vec{i} + \left(\frac{x}{x^2+y^2}\right)\vec{j} + z\,\vec{k}$.

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & z \end{vmatrix} = (0-0)\vec{i} - (0-0)\vec{j} + \left(\frac{(x^2+y^2)-x(2x)}{(x^2+y^2)^2} - \left(-\frac{(x^2+y^2)-y(2y)}{(x^2+y^2)^2}\right)\right)\vec{k} = 0$$

On the other hand, let $\vec{r}(\theta) = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$, where $0 \le \theta < 2\pi$.

$\vec{r}'(\theta) = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j}$; $\vec{F}(\vec{r}(\theta)) = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j}$.

$\vec{F}(\vec{r}(\theta))\cdot\vec{r}'(\theta) = (-\sin\theta\,\vec{i} + \cos\theta\,\vec{j})\cdot(-\sin\theta\,\vec{i} + \cos\theta\,\vec{j}) = \sin^2\theta + \cos^2\theta = 1$.

$\therefore \oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} d\theta = 2\pi \neq 0$

§16.8

Calculating Flux Using the Divergence Theorem

In Exercises 5–16, use the Divergence Theorem to find the outward flux of **F** across the boundary of the region *D*.

10. Cylindrical can $\mathbf{F} = (6x^2 + 2xy)\mathbf{i} + (2y + x^2z)\mathbf{j} + 4x^2y^3\mathbf{k}$

D: The region cut from the first octant by the cylinder $x^2 + y^2 = 4$ and the plane $z = 3$

Sol) $\vec{F}(x,y,z) = (6x^2+2xy)\vec{i} + (2y+x^2z)\vec{j} + (4x^2y^3)\vec{k}$.

$(\operatorname{div}\vec{F})(x,y,z) = \frac{\partial}{\partial x}(6x^2+2xy) + \frac{\partial}{\partial y}(2y+x^2z) + \frac{\partial}{\partial z}(4x^2y^3) = 12x+2y+2$.

Using cylindrical coordinates, $D = \{(r,\theta,z) \mid 0 \le r \le 2;\ 0 \le \theta \le \frac{\pi}{2};\ 0 \le z \le 3\}$

and $(\operatorname{div}\vec{F})(r,\theta,z) = 12r\cos\theta + 2r\sin\theta + 2$.

$\therefore$ By Divergence Theorem, $\iint_{\partial D} \vec{F}\cdot\vec{n}\,d\sigma = \iiint_D \nabla\cdot\vec{F}\,dV = \int_0^3\int_0^{\frac{\pi}{2}}\int_0^2 (12r\cos\theta + 2r\sin\theta + 2)(r\,dr\,d\theta\,dz)$

$= \left(\int_0^3 dz\right)\left[\left(\int_0^{\frac{\pi}{2}}\cos\theta\,d\theta\right)\left(\int_0^2 12r^2\,dr\right) + \left(\int_0^{\frac{\pi}{2}}\sin\theta\,d\theta\right)\left(\int_0^2 2r^2\,dr\right) + \left(\int_0^{\frac{\pi}{2}}d\theta\right)\left(\int_0^2 2r\,dr\right)\right]$

$= 3\cdot\left(1\cdot 32 + 1\cdot\frac{16}{3} + \frac{\pi}{2}\cdot 4\right) = 112 + 6\pi$ //

14. Thick sphere $\mathbf{F} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/\sqrt{x^2 + y^2 + z^2}$

D: The region $1 \le x^2 + y^2 + z^2 \le 4$

Sol) $\vec{F}(x,y,z) = \left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)\vec{i} + \left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)\vec{j} + \left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)\vec{k}.$

$$(\operatorname{div}\vec{F})(x,y,z) = \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial z}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \frac{1}{x^2+y^2+z^2}\left(\left(\left(\sqrt{x^2+y^2+z^2}\right)\cdot 1 - x\cdot\frac{x}{\sqrt{x^2+y^2+z^2}}\right) + \left(\left(\sqrt{x^2+y^2+z^2}\right)\cdot 1 - y\cdot\frac{y}{\sqrt{x^2+y^2+z^2}}\right) + \left(\left(\sqrt{x^2+y^2+z^2}\right)\cdot 1 - z\cdot\frac{z}{\sqrt{x^2+y^2+z^2}}\right)\right)$$

$$= \frac{1}{x^2+y^2+z^2}\left(3\sqrt{x^2+y^2+z^2} - \frac{x^2+y^2+z^2}{\sqrt{x^2+y^2+z^2}}\right) = \frac{2}{\sqrt{x^2+y^2+z^2}}$$

Using spherical coordinates, $D = \{(\rho,\phi,\theta) \mid 1 \le \rho \le 2 ; 0 \le \phi \le \pi ; 0 \le \theta < 2\pi\}$

and $(\operatorname{div}\vec{F})(\rho,\phi,\theta) = \frac{2}{\rho}$

$\therefore$ By Divergence Theorem, $\iint_{\partial D} \vec{F}\cdot\vec{n}\,d\sigma = \iiint_D \nabla\cdot\vec{F}\,dV = \int_0^{2\pi}\int_0^{\pi}\int_1^2 \left(\frac{2}{\rho}\right)(\rho^2\sin\phi\,d\rho\,d\phi\,d\theta)$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi}\sin\phi\,d\phi\right)\left(\int_1^2 2\rho\,d\rho\right) = 2\pi\cdot(2)\cdot(3) = 12\pi_{//}$$

16. Thick cylinder $\mathbf{F} = \ln(x^2+y^2)\mathbf{i} - \left(\frac{2z}{x}\tan^{-1}\frac{y}{x}\right)\mathbf{j} + z\sqrt{x^2+y^2}\,\mathbf{k}$

D: The thick-walled cylinder $1 \le x^2+y^2 \le 2, \quad -1 \le z \le 2$

Sol) $F(x,y,z) = (\ln(x^2+y^2))\vec{i} - \left(\frac{2z}{x}\tan^{-1}\left(\frac{y}{x}\right)\right)\vec{j} + \left(z\sqrt{x^2+y^2}\right)\vec{k}.$

$$(\operatorname{div}\vec{F})(x,y,z) = \frac{\partial}{\partial x}(\ln(x^2+y^2)) + \frac{\partial}{\partial y}\left(-\frac{2z}{x}\tan^{-1}\left(\frac{y}{x}\right)\right) + \frac{\partial}{\partial z}\left(z\sqrt{x^2+y^2}\right)$$

$$= \frac{2x}{x^2+y^2} - \frac{2z}{x}\cdot\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x} + \sqrt{x^2+y^2} = \frac{2x-2z}{x^2+y^2} + \sqrt{x^2+y^2}.$$

Using cylindrical coordinates, $D = \{(r,\theta,z) \mid 1 \le r \le \sqrt{2};\ 0 \le \theta < 2\pi;\ -1 \le z \le 2\}$

and $(\operatorname{div}\vec{F})(r,\theta,z) = \frac{2r\cos\theta - 2z}{r^2} + r$

$\therefore$ By Divergence Theorem, $\iint_{\partial D} \vec{F}\cdot\vec{n}\,d\sigma = \iiint_D \nabla\cdot\vec{F}\,dV = \int_{-1}^{2}\int_0^{2\pi}\int_1^{\sqrt{2}}\left(\frac{2r\cos\theta-2z}{r^2}+r\right)(r\,dr\,d\theta\,dz)$

$$= 2\left(\int_{-1}^{2}dz\right)\left(\int_0^{2\pi}\cos\theta\,d\theta\right)\left(\int_1^{\sqrt{2}}dr\right) - 2\left(\int_{-1}^{2}z\,dz\right)\left(\int_0^{2\pi}d\theta\right)\left(\int_1^{\sqrt{2}}\frac{1}{r}dr\right) + \left(\int_{-1}^{2}dz\right)\left(\int_0^{2\pi}d\theta\right)\left(\int_1^{\sqrt{2}}r^2dr\right)$$

$$= 0 - 2\cdot\frac{3}{2}\cdot 2\pi\cdot\ln\sqrt{2} + 3\cdot 2\pi\cdot\left(\frac{2\sqrt{2}-1}{3}\right) = 2\pi\left(-\frac{3}{2}\ln 2 + 2\sqrt{2}-1\right) //$$

Properties of Curl and Divergence

20. If $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a differentiable vector field, we define the notation $\mathbf{F} \cdot \nabla$ to mean

$$M\frac{\partial}{\partial x} + N\frac{\partial}{\partial y} + P\frac{\partial}{\partial z}.$$

For differentiable vector fields $\mathbf{F}_1$ and $\mathbf{F}_2$, verify the following identities.

a. $\nabla \times (\mathbf{F}_1 \times \mathbf{F}_2) = (\mathbf{F}_2 \cdot \nabla)\mathbf{F}_1 - (\mathbf{F}_1 \cdot \nabla)\mathbf{F}_2 + (\nabla \cdot \mathbf{F}_2)\mathbf{F}_1 - (\nabla \cdot \mathbf{F}_1)\mathbf{F}_2$

b. $\nabla(\mathbf{F}_1 \cdot \mathbf{F}_2) = (\mathbf{F}_1 \cdot \nabla)\mathbf{F}_2 + (\mathbf{F}_2 \cdot \nabla)\mathbf{F}_1 + \mathbf{F}_1 \times (\nabla \times \mathbf{F}_2) + \mathbf{F}_2 \times (\nabla \times \mathbf{F}_1)$

Sol) (a) Let $\vec{F}_1(x,y,z) = M^1(x,y,z)\vec{i} + N^1(x,y,z)\vec{j} + P^1(x,y,z)\vec{k}$

$\vec{F}_2(x,y,z) = M^2(x,y,z)\vec{i} + N^2(x,y,z)\vec{j} + P^2(x,y,z)\vec{k}$

$$(a)\ \vec{F}_1 \times \vec{F}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ M^1 & N^1 & P^1 \\ M^2 & N^2 & P^2 \end{vmatrix} = (N^1P^2 - N^2P^1)\vec{i} + (-M^1P^2 + M^2P^1)\vec{j} + (M^1N^2 - M^2N^1)\vec{k}$$

$$\text{LHS} = \nabla \times (\vec{F}_1 \times \vec{F}_2) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ N^1P^2 - N^2P^1 & -M^1P^2 + M^2P^1 & M^1N^2 - M^2N^1 \end{vmatrix}$$

$$= [(M^1N^2 - M^2N^1)_y - (-M^1P^2 + M^2P^1)_z]\vec{i}$$
$$- [(M^1N^2 - M^2N^1)_x - (N^1P^2 - N^2P^1)_z]\vec{j}$$
$$+ [(-M^1P^2 + M^2P^1)_x - (N^1P^2 - N^2P^1)_y]\vec{k}$$

$$= [M^1_yN^2 - M^2_yN^1 + M^1N^2_y - M^2N^1_y - (-M^1_zP^2 + M^2_zP^1) - (-M^1P^2_z + M^2P^1_z)]\vec{i}$$
$$- [M^1_xN^2 - M^2_xN^1 + M^1N^2_x - M^2N^1_x - (N^1_zP^2 - N^2_zP^1) - (N^1P^2_z - N^2P^1_z)]\vec{j}$$
$$+ [-M^1_xP^2 + M^2_xP^1 + -M^1P^2_x + M^2P^1_x - (N^1_yP^2 - N^2_yP^1) - (N^1P^2_y - N^2P^1_y)]\vec{k}$$

$$\text{RHS}: (\vec{F}_2 \cdot \nabla)(\vec{F}_1) = \left(M^2 \frac{\partial}{\partial x} + N^2 \frac{\partial}{\partial y} + P^2 \frac{\partial}{\partial z}\right)\left(M^1 \vec{i} + N^1 \vec{j} + P^1 \vec{k}\right)$$

$$= (M^2 M^1_x + N^2 M^1_y + P^2 M^1_z)\vec{i} + (M^2 N^1_x + N^2 N^1_y + P^2 N^1_z)\vec{j} + (M^2 P^1_x + N^2 P^1_y + P^2 P^1_z)\vec{k}.$$

$$(\vec{F}_1 \cdot \nabla)(\vec{F}_2) = \left(M^1 \frac{\partial}{\partial x} + N^1 \frac{\partial}{\partial y} + P^1 \frac{\partial}{\partial z}\right)\left(M^2 \vec{i} + N^2 \vec{j} + P^2 \vec{k}\right)$$

$$= (M^1 M^2_x + N^1 M^2_y + P^1 M^2_z)\vec{i} + (M^1 N^2_x + N^1 N^2_y + P^1 N^2_z)\vec{j} + (M^1 P^2_x + N^1 P^2_y + P^1 P^2_z)\vec{k}.$$

$$(\nabla \cdot \vec{F}_2)\vec{F}_1 = (M^2_x + N^2_y + P^2_z)(M^1 \vec{i} + N^1 \vec{j} + P^1 \vec{k})$$

$$= (M^2_x M^1 + N^2_y M^1 + P^2_z M^1)\vec{i} + (M^2_x N^1 + N^2_y N^1 + P^2_z N^1)\vec{j} + (M^2_x P^1 + N^2_y P^1 + P^2_z P^1)\vec{k}$$

$$(\nabla \cdot \vec{F}_1)\vec{F}_2 = (M^1_x + N^1_y + P^1_z)(M^2 \vec{i} + N^2 \vec{j} + P^2 \vec{k})$$

$$= (M^1_x M^2 + N^1_y M^2 + P^1_z M^2)\vec{i} + (M^1_x N^2 + N^1_y N^2 + P^1_z N^2)\vec{j} + (M^1_x P^2 + N^1_y P^2 + P^1_z P^2)\vec{k}$$

$$\therefore \text{RHS} = (\vec{F}_2 \cdot \nabla)(\vec{F}_1) - (\vec{F}_1 \cdot \nabla)(\vec{F}_2) + (\nabla \cdot \vec{F}_2)\vec{F}_1 - (\nabla \cdot \vec{F}_1)\vec{F}_2$$

$$\begin{aligned}
= &\left[(\cancel{M^2 M^1_x} + N^2 M^1_y + P^2 M^1_z)\vec{i} + (M^2 N^1_x + \cancel{N^2 N^1_y} + P^2 N^1_z)\vec{j} + (M^2 P^1_x + N^2 P^1_y + \cancel{P^2 P^1_z})\vec{k}\right] \\
&- \left[(\cancel{M^1 M^2_x} + N^1 M^2_y + P^1 M^2_z)\vec{i} + (M^1 N^2_x + \cancel{N^1 N^2_y} + P^1 N^2_z)\vec{j} + (M^1 P^2_x + N^1 P^2_y + \cancel{P^1 P^2_z})\vec{k}\right] \\
&+ \left[(\cancel{M^2_x M^1} + N^2_y M^1 + P^2_z M^1)\vec{i} + (M^2_x N^1 + \cancel{N^2_y N^1} + P^2_z N^1)\vec{j} + (M^2_x P^1 + N^2_y P^1 + \cancel{P^2_z P^1})\vec{k}\right] \\
&- \left[(\cancel{M^1_x M^2} + N^1_y M^2 + P^1_z M^2)\vec{i} + (M^1_x N^2 + \cancel{N^1_y N^2} + P^1_z N^2)\vec{j} + (M^1_x P^2 + N^1_y P^2 + \cancel{P^1_z P^2})\vec{k}\right]
\end{aligned}$$

$$\begin{aligned}
&\left[\boxed{M^1_y N^2 - M^2_y N^1 + M^1 N^2_y - M^2 N^1_y} - \boxed{(-M^1_z P^2 + M^2_z P^1) - (-M^1 P^2_z + M^2 P^1_z)}\right]\vec{i} \\
= -&\left[M^1_x N^2 - M^2_x N^1 + M^1 N^2_x - M^2 N^1_x - (N^1_z P^2 - N^2_z P^1) \quad - (N^1 P^2_z - N^2 P^1_z)\right]\vec{j} \qquad = \text{LHS}_{/\!/} \\
+ &\left[-M^1_x P^2 + M^2_x P^1 + -M^1 P^2_x + M^2 P^1_x - (N^1_y P^2 - N^2_y P^1) \quad - (N^1 P^2_y - N^2 P^1_y)\right]\vec{k}
\end{aligned}$$

(b) LHS $= \nabla\left(\left(M^1\vec{i} + N^1\vec{j} + P^1\vec{k}\right)\cdot\left(M^2\vec{i} + N^2\vec{j} + P^2\vec{k}\right)\right) = \nabla\left(M^1M^2 + N^1N^2 + P^1P^2\right)$

$= \left(M^1M^2 + N^1N^2 + P^1P^2\right)_x \vec{i} + \left(M^1M^2 + N^1N^2 + P^1P^2\right)_y \vec{j} + \left(M^1M^2 + N^1N^2 + P^1P^2\right)_z \vec{k}$

$$\begin{aligned}
= &\left(M^1_xM^2 + N^1_xN^2 + P^1_xP^2 + M^1M^2_x + N^1N^2_x + P^1P^2_x\right)\vec{i} \\
+ &\left(M^1_yM^2 + N^1_yN^2 + P^1_yP^2 + M^1M^2_y + N^1N^2_y + P^1P^2_y\right)\vec{j} \\
+ &\left(M^1_zM^2 + N^1_zN^2 + P^1_zP^2 + M^1M^2_z + N^1N^2_z + P^1P^2_z\right)\vec{k}
\end{aligned}$$

RHS: $(\vec{F}_1\cdot\nabla)(\vec{F}_2) = \left(M^1\frac{\partial}{\partial x} + N^1\frac{\partial}{\partial y} + P^1\frac{\partial}{\partial z}\right)\left(M^2\vec{i} + N^2\vec{j} + P^2\vec{k}\right)$

$= \left(M^1M^2_x + N^1M^2_y + P^1M^2_z\right)\vec{i} + \left(M^1N^2_x + N^1N^2_y + P^1N^2_z\right)\vec{j} + \left(M^1P^2_x + N^1P^2_y + P^1P^2_z\right)\vec{k}.$

$(\vec{F}_2\cdot\nabla)(\vec{F}_1) = \left(M^2\frac{\partial}{\partial x} + N^2\frac{\partial}{\partial y} + P^2\frac{\partial}{\partial z}\right)\left(M^1\vec{i} + N^1\vec{j} + P^1\vec{k}\right)$

$= \left(M^2M^1_x + N^2M^1_y + P^2M^1_z\right)\vec{i} + \left(M^2N^1_x + N^2N^1_y + P^2N^1_z\right)\vec{j} + \left(M^2P^1_x + N^2P^1_y + P^2P^1_z\right)\vec{k}.$

$$\nabla\times\vec{F}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M^2 & N^2 & P^2 \end{vmatrix} = \left(P^2_y - N^2_z\right)\vec{i} + \left(-P^2_x + M^2_z\right)\vec{j} + \left(N^2_x - M^2_y\right)\vec{k}$$

$$\vec{F}_1\times\left(\nabla\times\vec{F}_2\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ M^1 & N^1 & P^1 \\ P^2_y - N^2_z & -P^2_x + M^2_z & N^2_x - M^2_y \end{vmatrix} = \left(N^1N^2_x - N^1M^2_y + P^1P^2_x - P^1M^2_z\right)\vec{i}$$

$- \left(M^1N^2_x - M^1M^2_y - P^1P^2_y + P^1N^2_z\right)\vec{j} + \left(-M^1P^2_x + M^1M^2_z - N^1P^2_y + N^1N^2_z\right)\vec{k}$

$$\vec{F}_2\times\left(\nabla\times\vec{F}_1\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ M^2 & N^2 & P^2 \\ P^1_y - N^1_z & -P^1_x + M^1_z & N^1_x - M^1_y \end{vmatrix} = \left(N^2N^1_x - N^2M^1_y + P^2P^1_x - P^2M^1_z\right)\vec{i}$$

$- \left(M^2N^1_x - M^2M^1_y - P^2P^1_y + P^2N^1_z\right)\vec{j} + \left(-M^2P^1_x + M^2M^1_z - N^2P^1_y + N^2N^1_z\right)\vec{k}$

$$\therefore \text{RHS} = (\vec{F}_1 \cdot \nabla)(\vec{F}_2) + (\vec{F}_2 \cdot \nabla)(\vec{F}_1) + \vec{F}_1 \times (\nabla \times \vec{F}_2) + \vec{F}_2 \times (\nabla \times \vec{F}_1)$$

$$= (M^1 M^2_x + \cancel{N^1 M^2_y} + \cancel{P^1 M^2_z})\vec{i} + (\cancel{M^1 N^2_x} + N^1 N^2_y + \cancel{P^1 N^2_z})\vec{j} + (\cancel{M^1 P^2_x} + \cancel{N^1 P^2_y} + P^1 P^2_z)\vec{k}.$$

$$+ (M^2 M^1_x + \cancel{N^2 M^1_y} + \cancel{P^2 M^1_z})\vec{i} + (\cancel{M^2 N^1_x} + N^2 N^1_y + \cancel{P^2 N^1_z})\vec{j} + (\cancel{M^2 P^1_x} + \cancel{N^2 P^1_y} + P^2 P^1_z)\vec{k}.$$

$$+ \left[(N^1 N^2_x - \cancel{N^1 M^2_y} + P^1 P^2_x - \cancel{P^1 M^2_z})\vec{i} - (\cancel{M^1 N^2_x} - M^1 M^2_y - P^1 P^2_y + \cancel{P^1 N^2_z})\vec{j} + (-\cancel{M^1 P^2_x} + M^1 M^2_z - \cancel{N^1 P^2_y} + N^1 N^2_z)\vec{k} \right]$$

$$+ \left[(N^2 N^1_x - \cancel{N^2 M^1_y} + P^2 P^1_x - \cancel{P^2 M^1_z})\vec{i} - (\cancel{M^2 N^1_x} - M^2 M^1_y - P^2 P^1_y + \cancel{P^2 N^1_z})\vec{j} + (-\cancel{M^2 P^1_x} + M^2 M^1_z - \cancel{N^2 P^1_y} + N^2 N^1_z)\vec{k} \right]$$

$$= (M^1_x M^2 + N^1_x N^2 + P^1_x P^2 + M^1 M^2_x + N^1 N^2_x + P^1 P^2_x)\vec{i} \quad = \text{LHS}.$$
$$+ (M^1_y M^2 + N^1_y N^2 + P^1_y P^2 + M^1 M^2_y + N^1 N^2_y + P^1 P^2_y)\vec{j}$$
$$+ (M^1_z M^2 + N^1_z N^2 + P^1_z P^2 + M^1 M^2_z + N^1 N^2_z + P^1 P^2_z)\vec{k}$$

Theory and Examples

23. **a.** Show that the outward flux of the position vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ through a smooth closed surface S is three times the volume of the region enclosed by the surface.

b. Let $\mathbf{n}$ be the outward unit normal vector field on S. Show that it is not possible for $\mathbf{F}$ to be orthogonal to $\mathbf{n}$ at every point of S.

Sol) (a) $F(x,y,z) = x\vec{i} + y\vec{j} + z\vec{k}$.

$(\mathrm{div}\,\vec{F})(x,y,z) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$.

$\therefore$ By Divergence Theorem, $\iint_S \vec{F}\cdot\vec{n}\,d\sigma = \iiint_D \nabla\cdot\vec{F}\,dV$, where D is the region bounded by S.

$$= 3\cdot\iiint_D dV = 3\cdot \mathrm{Vol}(D)_{//}$$

(b) Suppose on the contrary, $\vec{F}$ is orthogonal to $\vec{n}$ at every point of S.

Then $(\vec{F}\cdot\vec{n})(x,y,z) = 0$, for any $(x,y,z)\in S$.

$\therefore \iint_S \vec{F}\cdot\vec{n}\,d\sigma = \iint_S 0\,d\sigma = 0$

On the other hand, by (a), $\iint_S \vec{F}\cdot\vec{n}\,d\sigma = 3\cdot\mathrm{Vol}(D) \neq 0$

$\therefore$ Contradiction arises. Therefore, $\vec{F}$ cannot be orthogonal to $\vec{n}$ at every point of S.

27. Harmonic functions A function $f(x, y, z)$ is said to be *harmonic* in a region D in space if it satisfies the Laplace equation

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

throughout D.

a. Suppose that f is harmonic throughout a bounded region D enclosed by a smooth surface S and that $\mathbf{n}$ is the chosen unit normal vector on S. Show that the integral over S of $\nabla f \cdot \mathbf{n}$, the derivative of f in the direction of $\mathbf{n}$, is zero.

b. Show that if f is harmonic on D, then

$$\iint_S f \nabla f \cdot \mathbf{n} \, d\sigma = \iiint_D |\nabla f|^2 \, dV.$$

Sol) (a) Since f is harmonic, $\operatorname{div}(\nabla f) = 0$.

$\therefore$ By Divergence Theorem, $\iint_S (\nabla f) \cdot \vec{n} \, d\sigma = \iiint_D \operatorname{div}(\nabla f) \, dV = \iiint_D 0 \, dV = 0$.

(b) By Divergence Theorem, $\iint_S (f \nabla f) \cdot \vec{n} \, d\sigma = \iiint_D \operatorname{div}(f \nabla f) \, dV$

where $\operatorname{div}(f \nabla f) = \operatorname{div}\left(f \cdot \frac{\partial f}{\partial x} \vec{i} + f \cdot \frac{\partial f}{\partial y} \vec{j} + f \cdot \frac{\partial f}{\partial z} \vec{k}\right)$

$= \frac{\partial}{\partial x}\left(f \cdot \frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(f \cdot \frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(f \cdot \frac{\partial f}{\partial z}\right)$

$= f \cdot \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) + \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2\right]$

$= 0 + |\nabla f|^2$ (since f is harmonic)

$= |\nabla f|^2$

$\therefore \iint_S (f \nabla f) \cdot \vec{n} \, d\sigma = \iiint_D \operatorname{div}(f \nabla f) \, dV = \iiint_D |\nabla f|^2 \, dV$ //